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Numerical solution of the oxygen diffusion problem in cylindrically shaped sections of tissue

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SUMMARY

A mathematical model is presented which describes the diffusion of oxygen in absorbing tissue, and numerical solution of its partial differential equation is obtained by the finite difference equations. The diffusion with absorption model is associated with the process of a moving boundary which marks the furthest penetration of oxygen in the absorbing cylindrically shaped sections of tissue and also allows for an initial distribution of oxygen through the absorbing tissue. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

A problem arises from the diffusion of oxygen where some of the oxygen is absorbed and thereby removed from the diffusion process. A moving boundary is an essential feature of this problem but the conditions that determine its movements are different, the concentration of oxygen always being zero at the boundary with no oxygen diffusing across the boundary at any time. The problem of oxygen diffusion in an absorbing tissue was first discussed by Crank and Gupta [1], they used an explicit finite difference formulation of the governing differential equation for a one-dimensional problem. Near the moving boundary, a three-point Lagrange interpolation formula was used to solve the oxygen concentration, and the location of the point on the moving boundary was determined by a Taylor series about the moving boundary. For small values of time, an approximate expression was used to compute concentration due to the difficulties caused by the discontinuous boundary conditions on the sealed surface. Crank and Gupta [2] also proposed a uniform space grid

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(in one dimension). Since then the problem has attracted a good deal of attention from a number of authors who have attempted it by various methods, Hansen and Haugaard [3] proposed an integral equation formulation of the diffusion–absorption problem. Ferris and Hill [4] fixed the boundary by appropriate transformation, Berger *et al.* [5] proposed a two-dimensional fixed domain model for solving diffusion absorption problems (truncation method). Baiocchi and Pozzi [6] proposed a variational inequality approach to the diffusion absorption problem, while Miller *et al.* [7] dealt with using finite elements. Çatal [8] traced the moving boundary using the constrained integral method, the profile of moving boundary is determined by the third and the fourth-order polynomials. Ahmed [9] proposed a numerical method to form a double linear system of equations, each of dimension (4×4), the first system is formed through applying first, second, third and fourth moments and using assumed profile for the concentration containing four unknown functions. The second system is formed through applying the boundary conditions given in addition to another assumed condition, that is, the concentration at $x=0$ is an unknown function of time. The result of the first system becomes an entry data for the second is leading to the concentration at a fixed surface $x=0$. More recently a paper dealing with this problem was developed by Boureghda [10]. In this paper the author presented an approximate analytical solution using Fourier expansion and the technique of Crank and Gupta is presented with some modification and numerical solutions of its partial differential equation are obtained, which describe the diffusion of oxygen in an absorbing tissue. More references to this problem may be found in the thesis *Numerical Methods for Solving one Dimensional Problems with a Moving Boundary* by Boureghda [11] and in References [12, 13]. There is yet another class of methods which are known as variable time step methods, examples of such methods are due to Douglas and Gallie [14], Goodling and Khader [15], Gupta and Kumar [16]. These methods have been successfully employed for solving moving boundary problems involving change of phase due to heat conduction, but we cannot apply them here as an explicit relationship between velocity of the moving boundary and the heat flux is absent in the present problem. In the present paper we describe the diffusion of oxygen in absorbing cylindrically shaped sections of tissue using the development of Taylor series near the moving boundary in space direction.

2. CYLINDRICAL PROBLEM

A general version of the oxygen problem in cylindrically shaped sections of tissue is described by Galib *et al.* [17]. They assumed the following analytic expression for c which satisfies the tissue boundary conditions and which is zero and has a zero normal flux on the moving boundary.

$$c(x,\theta,t)=0.5[\rho(\theta,t)-r]^2-\frac{[\rho(\theta,t)-r]^3}{3[\rho(\theta,t)-r_i]} \tag{1}$$

and in non-dimensional terms the mathematical problem in two dimensions (r, and θ) and time, in cylindrical coordinates is defined by the equations

$$\frac{\partial c}{\partial t}=\frac{\partial^2 c}{\partial x^2}+\frac{\partial c}{x\partial x}+\frac{\partial^2 c}{x^2\partial\theta^2}+f(x,\theta,t,c) \tag{2}$$

where $c(x,\theta,t)$ is the oxygen concentration in the tissue, x is the radial coordinate, θ is the angular coordinate, t is the time and $f(x,\theta,t,c)$ is the rate of absorption of oxygen. $\rho(\theta,t)$ the position of

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the moving boundary, r_i is the sealed surface radius and r_0 is the outer tissue radius. The boundary radius conditions are

$$c(\rho, \theta, t) = 0 \tag{3}$$

$$\frac{\partial c(\rho, \theta, t)}{\partial r} = 0 \tag{4}$$

$$\frac{\partial c(\rho, \theta, t)}{\partial \theta} = 0 \tag{5}$$

$$\frac{\partial c(r_i, \theta, t)}{\partial r} = 0 \tag{6}$$

and

$$c(r_0, \theta, t) = 0 \tag{7}$$

The absorption function is found by substituting this expression for c into the partial differential equation (2). For numerical solution, the finite difference equations are formulated using the Crank–Nicolson method.

The one-dimensional oxygen diffusion problem solved by Boureghda [10] can be posed in terms of the cylindrical coordinates.

Assuming that the concentration c is independent of θ, the equation is

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} + \frac{\partial c}{x \partial x} - 1 \tag{8}$$

Taking the boundary to be the unit cylinder ($x = 1$), the boundary conditions are:
the boundary of the cylinder ($x = 1$)

$$\frac{\partial c}{\partial x} = 0, \quad t \geqslant 0 \tag{9}$$

and on the moving boundary ($x = s(t)$)

$$c = \frac{\partial c}{\partial x} = 0, \quad t \geqslant 0 \tag{10}$$

initially ($t = 0$)
the moving boundary is at $x = x1 = s(0)$ and c satisfies

$$\frac{\mathrm{d}^2 c}{\mathrm{d}x^2} + \frac{\mathrm{d}c}{x\,\mathrm{d}x} - 1 = 0 \tag{11}$$

This has the solution

$$c = 0.25x^2 + A \ \ln(x) + B \tag{12}$$

and the boundary conditions ($x = 1, x = x_1$) give

$$A = -0.5x_1^2, \quad B = 0.5(x_1^2)\ln(x_1) - 0.25x_1^2$$

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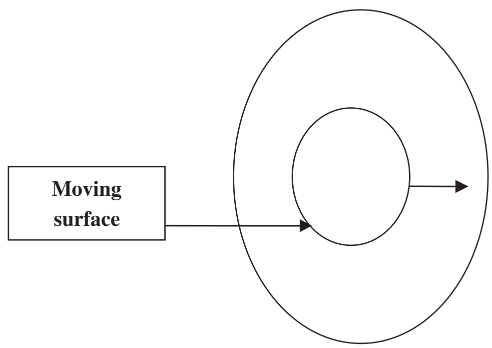

Figure 1. Inside problem.

so that

$$c = 0.25(x^2 - x_1^2) - 0.5x_1^2(\ln(x) - \ln(x_1)) \tag{13}$$

Note that from this initial solution

$$\frac{\partial c}{\partial x} = 0.5x - 0.5\left(\frac{x_1^2}{x}\right) \tag{14}$$

at the surface of the cylinder ($x = 1$)

$$c = 0.25(1 - x_1^2) + 0.5x_1^2 \ln(x_1) \tag{15}$$

and

$$\frac{\partial c}{\partial x} = 0.5(1 - x_1^2) \tag{16}$$

Because of the condition

$$\frac{\partial c}{\partial x} = 0, \quad x = 1, \quad t > 0 \tag{17}$$

the solution has therefore a singularity at the point $x = 1$, $t = 0$, as well as a moving boundary.

Finally, there are two different problems to consider. The oxygen may be outside the cylinder and diffusing into the medium inside the cylinder where it is absorbed. Thus, the equations hold inside the cylinder. Alternatively, the oxygen may be inside the cylinder and diffusing into the surrounding medium as in Galib *et al*. [17]. In this case, the problem is solved for outside the cylinder. These two cases will be considered separately (see Figures 1 and 2).

3. OXYGEN DIFFUSION INSIDE THE CYLINDER

The problem involves the solution of the equation

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} + \frac{\partial c}{x \partial x} - 1, \quad t \geqslant 0, \quad s(t) \leqslant x < 1 \tag{18}$$

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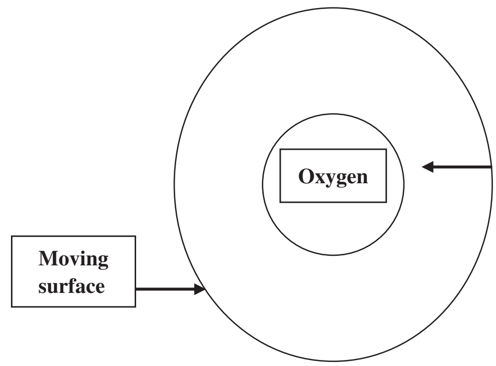

Figure 2. Outside problem.

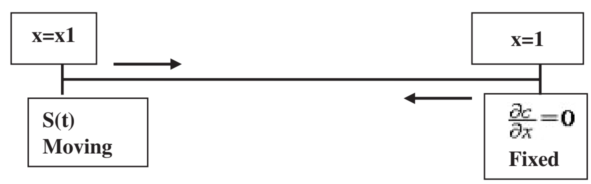

Figure 3. Inside problem.

with

$$c=\frac{\partial c}{\partial x}=0, \quad x=s(t), \quad t\geqslant 0, \quad s(0)=x_1 \tag{19}$$

$$\frac{\partial c}{\partial x}=0, \quad x=1, \quad t\geqslant 0 \tag{20}$$

and

$$c=0.25(x^2-x_1^2)-0.5x_1^2(\ln(x)-\ln(x_1)), \quad x_1\leqslant x<1, \quad t=0 \tag{21}$$

The problem has a singularity at the point $(x=1, t=0)$ so that an approximate analytical solution must be found for the first values of t (see Figure 3).

4. APPROXIMATE ANALYTICAL SOLUTION

It is known (see e.g. Gray and Macrobert [18]) that solutions of homogeneous partial differential equation

$$\frac{\partial c}{\partial t}=\frac{\partial^2 c}{\partial x^2}+\frac{\partial c}{x\partial x} \tag{22}$$

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are of the form

$$c = \mathrm{e}^{-\lambda^2 t}(C J_0(\lambda x) + D Y_0(\lambda x)) \tag{23}$$

where $J_0(\lambda x)$, $Y_0(\lambda x)$ are Bessel function of order zero. This suggests that we should look for a series solution of the form

$$c = a_0(t) + \sum_s a_s(t)(C J_0(\lambda_s x) + D Y_0(\lambda_s x)) \tag{24}$$

writing

$$u(x) = C J_0(\lambda x) + D Y_0(\lambda x) \tag{25}$$

we note that u satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{x \partial x} + \lambda^2 u = 0 \tag{26}$$

hence

$$\frac{\partial c}{\partial t} = \frac{\mathrm{d}a_0(t)}{\mathrm{d}t} + \sum_{s=1} \frac{\mathrm{d}a_s(t)}{\mathrm{d}t} u \tag{27}$$

$$\frac{\partial^2 c}{\partial x^2} + \frac{\partial c}{x \partial x} = \sum_s a_s \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{x \partial x} \right) = \sum_s - a_s \lambda_s^2 u_s \tag{28}$$

equating coefficients in (8), we obtain

$$\frac{\mathrm{d}a_0}{\mathrm{d}t} = -1$$

$$\frac{\mathrm{d}a_s}{\mathrm{d}t} = -a_s \lambda_s^2, \quad s \geqslant 1$$

giving

$$a_0 = A_0 - t$$

$$a_s = A_s \mathrm{e}^{-\lambda_s^2 t}$$

and

$$c = A_0 - t + \sum_s A_s \mathrm{e}^{-\lambda_s^2 t}(J_0(\lambda_s x) + K_s Y_0(\lambda_s x)), \quad k_s = \frac{D}{C} \tag{29}$$

when $t = 0$, we have

$$c(0, x) = 0.25(x^2 - x_1^2) - 0.5 x_1^2 (\ln(x) - \ln(x_1)) \equiv f(x) \tag{30}$$

so that we require the Fourier Bessel Expansion

$$f(x) = A_0 + \sum_s A_s (C J_0(\lambda_s x) + D Y_0(\lambda_s x)) \tag{31}$$

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For the inside problem, c remains finite as $x \to 0$ so that $k_s = 0$ and we can consider the expansion

$$f(x) = A_0 + \sum_s A_s J_0(\lambda_s x) \tag{32}$$

4.1. Expansion in terms of Bessel functions

If λ, μ are constant, then

$$(\mu^2 - \lambda^2) \int_0^1 x J_0(\lambda x) J_0(\mu x)\, \mathrm{d}x = \lambda J_0'(\lambda) J_0(\mu) - \mu J_0'(\mu) J_0(\lambda) \tag{33}$$

and

$$\int_0^1 x J_0^2(\lambda x)\, \mathrm{d}x = 0.5[J_0^2(\lambda) + J_0'^2(\lambda)] \tag{34}$$

Hence, for the expansion

$$f(x) = A_0 + \sum_s A_s J_0(\lambda_s x) \tag{35}$$

on multiplying by x, if $s = 0$ or $x J_0(\lambda_s x)$ otherwise and integrating over the range 0, 1 if $J_0(\lambda_s) = 0$ (the usual case) we obtain $A_0 = 0$

$$A_s = \frac{\int_0^1 x f(x) J_0(\lambda_s x)\, \mathrm{d}x}{\int_0^1 x J_0^2(\lambda_s x)\, \mathrm{d}x} \tag{36}$$

$$2\, \frac{\int_0^1 x f(x) J_0(\lambda_s x)\, \mathrm{d}x}{J_1^2(\lambda_s)}, \quad s \geqslant 1 \tag{37}$$

whereas if $J_0'(\lambda_s) = 0$ we obtain

$$A_0 = \frac{(\int_0^1 x f(x)\, \mathrm{d}x)}{(\int_0^1 x\, \mathrm{d}x)} = 2 \int_0^1 x f(x)\, \mathrm{d}x \tag{38}$$

$$A_s = \frac{\int_0^1 x f(x) J_0(\lambda_s x)\, \mathrm{d}x}{\int_0^1 x J_0^2(\lambda_s x)\, \mathrm{d}x} = 2 \frac{\int_0^1 x f(x) J_0(\lambda_s x)\, \mathrm{d}x}{J_0^2(\lambda_s)}, \quad s \geqslant 1 \tag{39}$$

we require to find the Fourier Bessel expansion of

$$f(x) = 0.25(x^2 - x_1^2) - 0.5 x_1^2(\ln(x) - \ln(x_1)) \tag{40}$$

4.2. Bessel function integrals

Given integrals

$$\int x^m J_0(x)\, \mathrm{d}x = x^m J_1(x) - (m-1) \int x^{(m-1)} J_1(x)\, \mathrm{d}x$$

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$$= x^m J_1(x) + (m-1)\int x^{(m-1)} J_0'(x)\,\mathrm{d}x$$

$$= x^m J_1(x) + (m-1)x^{(m-1)} J_0(x) - (m-1)^2 \int x^{(m-2)} J_0(x)\,\mathrm{d}x$$

$$\int x J_0(x)\,\mathrm{d}x = x J_1(x)$$

so that

$$\int x^3 J_0(x)\,\mathrm{d}x = x^3 J_1(x) + 2x^2 J_0(x) - 4x J_1(x)$$

$$= (x^3 - 4x) J_1(x) + 2x^2 J_0(x)$$

$$\int x J_0(x)\ \ln(x)\,\mathrm{d}x = x J_1(x)\ \ln(x) + J_0(x)$$

4.3. Some algebra

After some algebra, we find that $\{\lambda$ is root of $J_1(\lambda) = 0\}$

$$\lambda^2 \int_0^1 x J_0(\lambda x)\,\mathrm{d}x = \lambda J_1(\lambda)$$

$$\lambda^4 \int_0^1 x^3 J_0(\lambda x)\,\mathrm{d}x = (\lambda^3 - 4\lambda) J_1(\lambda) + 2\lambda^2 J_0(\lambda)$$

and

$$\lambda^2 \int_0^1 x J_0(\lambda x)\ln(x)\,\mathrm{d}x = J_0(\lambda) - 1$$

so that

$$A_s = \frac{[(1 - x_1^2) J_0(\lambda_s) + x_1^2]}{\lambda_s^2 J_0^2(\lambda_s)} \tag{41}$$

and

$$A_0 = 0.125 + 0.5x_1^2\ \ln(x_1) \tag{42}$$

5. OXYGEN DIFFUSION OUTSIDE THE CYLINDER

The problem is to solve

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} + \frac{\partial c}{x\,\partial x} - 1, \quad t \geqslant 0, \quad (1 \leqslant x \leqslant x_1), \quad s(t) \leqslant x_1 \tag{43}$$

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with

$$c=\frac{\partial c}{\partial x}=0, \quad x=s(t), \quad t\geqslant 0, \quad s(0)=x_1=2 \tag{44}$$

$$\frac{\partial c}{\partial x}=0, \quad x=1, \quad t\geqslant 0 \tag{45}$$

and

$$c=0.25(x^2-x_1^2)-0.5x_1^2(\ln(x)-\ln(x_1)), \quad 1\leqslant x<x_1=2, \quad t=0 \text{ (see Figure 4)} \tag{46}$$

5.1. *Approximate analytical solution*

We assume series solution of the following form:

$$c(x,t)=a_0(t)+\sum_s a_s(t)(C_s J_0(\lambda_s x)+D_s Y_0(\lambda_s x)) \tag{47}$$

where $J_0(\lambda_s x)$, $Y_0(\lambda_s x)$ are Bessel functions of order zero. The solution has to satisfy the boundary conditions.

In order to satisfy the boundary condition at $x=1$, we require

$$C_s J_0'(\lambda_s x)+D_s Y_0'(\lambda_s x)=0 \tag{48}$$

i.e. assume the expansion of the following form:

$$c(x,t)=a_0(t)+\sum_s a_s(t)\left\{\frac{J_0(\lambda_s x)}{J_0'(\lambda_s)}-\frac{Y_0(\lambda_s x)}{Y_0'(\lambda_s)}\right\} \tag{49}$$

To satisfy the boundary condition at $x=x_1$, the values of λ_s are given by

$$\frac{J_0(\lambda_s x_1)}{J_0'(\lambda_s)}-\frac{Y_0(\lambda_s x_1)}{Y_0'(\lambda_s)}=0 \tag{50}$$

The coefficients $a_s(t)$ are given as before, i.e.

$$a_0=A_0-t$$

and

$$a_s=A_s \mathrm{e}^{-\lambda_s^2 t}$$

In this case,

$$A_0=0$$

$$A_s=\frac{P_s}{Q_s}, \quad s\geqslant 1$$

where

$$P_s=\int_1^{x_1} x f(x)\left\{\frac{J_0(\lambda_s x)}{J_0'(\lambda_s)}-\frac{Y_0(\lambda_s x)}{Y_0'(\lambda_s)}\right\}\mathrm{d}x \tag{51}$$

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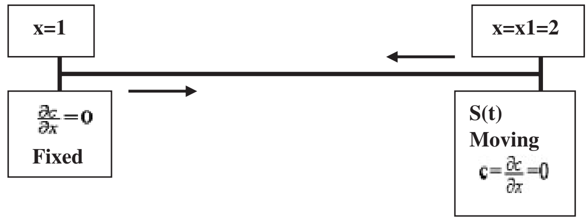

Figure 4. Outside problem.

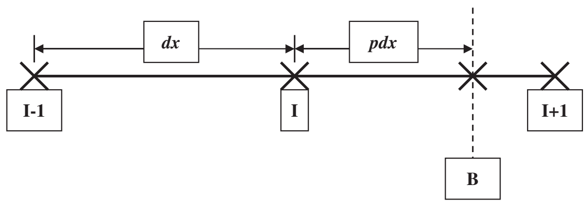

Figure 5. Fixed grid.

$$Q_s=\int_1^{x_1} x\left\{\frac{J_0(\lambda_s x)}{J_0'(\lambda_s)}-\frac{Y_0(\lambda_s x)}{Y_0'(\lambda_s)}\right\}^2 \mathrm{d}x = 0.5\left\{\begin{array}{c} x_1^2\left[\dfrac{J_0'(\lambda_s x_1)}{J_0'(\lambda_s)}-\dfrac{Y_0'(\lambda_s x_1)}{Y_0'(\lambda_s)}\right]^2 \\ -\left[\dfrac{J_0(\lambda_s)}{J_0'(\lambda_s)}-\dfrac{Y_0(\lambda_s)}{Y_0'(\lambda_s)}\right]^2\end{array}\right\} \mathrm{d}x \tag{52}$$

Using the expressions for the integrals in Section 4.2 which are also true for Y_0, we obtain

$$A_s=\frac{\left\{\dfrac{-x_1\left[\dfrac{J_1(\lambda_s x_1)}{J_0'(\lambda_s)}-\dfrac{Y_1(\lambda_s x_1)}{Y_0'(\lambda_s)}\right]}{\lambda_s^3}+0.5(x_1^2-1)\dfrac{\left[\dfrac{J_0(\lambda_s)}{J_0'(\lambda_s)}-\dfrac{Y_0(\lambda_s)}{Y_0'(\lambda_s)}\right]}{\lambda_s^2}\right\}}{Q_S}, \quad S\geqslant 1 \tag{53}$$

For the approximate analytical solution, we use the same method as mentioned for the inside problem, but the boundary conditions are different (see Figure 4).

Here in the numerical solution, for the interval 0.1, we have solved the problem using the approximate analytical solution for the first three steps, but for interval 0.05 we obtain the initial values from formula (46). See numerical results in Tables IV–VI.

6. NUMERICAL SOLUTION OF CYLINDRICAL PROBLEMS

For the numerical solution to problem (8)–(13), we use the development of Taylor series near the moving boundary in space direction as in [10], where Crank used Lagrange-type formula in the cartesian problem (see Figure 5).

Table I. The values of $10^6 c$ and the position of the moving boundary for inside cylindrical problem $\delta t = 0.001$ and $\delta x = 0.1$.

	x					
t	0.1	0.3	0.5	0.7	0.9	Moving boundary
0.005	188965	116893	058499	019410	000399	0.07176
0.010	172151	110761	054487	016398	000000	0.10196
0.015	156537	103834	050423	013533	000000	0.12371
0.025	129101	088634	041966	008604	000000	0.16425
0.030	116860	080831	037592	006532	000000	0.18388
0.035	105393	073081	033174	004576	000000	0.20434
0.040	094593	065457	028750	002798	000000	0.22520
0.045	084378	058007	024350	001388	000000	0.24731
0.050	074685	050759	020034	000394	000000	0.27729
0.060	056675	036942	011819	000000	000000	0.32937
0.070	040266	024103	004445	000000	000000	0.40571
0.080	025266	012369	000000	000000	000000	0.50460
0.095	005143	000000	000000	000000	000000	0.74950
0.097	002166	000000	000000	000000	000000	0.83418

Table II. The values of $10^6 c$ and the position of the moving boundary for inside cylindrical problem $\delta t = 0.001$ and $\delta x = 0.05$.

	x					
t	0.1	0.3	0.5	0.7	0.9	Moving boundary
0.005	183472	114633	057492	019100	000503	0.06828
0.010	162652	106403	052518	015758	000000	0.09876
0.015	144169	097367	047548	012638	000000	0.12390
0.025	112728	078666	037416	007251	000000	0.17126
0.030	099105	069536	032299	004989	000000	0.19551
0.035	086592	060721	027227	003011	000000	0.21977
0.040	075041	052271	022261	001396	000000	0.24715
0.045	064334	044215	017461	000284	000000	0.27618
0.050	054381	036561	012883	000000	000000	0.30960
0.060	036446	022476	004746	000000	000000	0.39234
0.070	020781	010075	000000	000000	000000	0.50390
0.080	007097	000000	000000	000000	000000	0.68798
0.083	003371	000000	000000	000000	000000	0.78603
0.085	000000	000000	000000	000000	000000	0.89570

We obtain

$$\frac{\partial^2 c}{\partial x^2} = \left(\frac{2}{(\delta x)}\right)\left[\frac{c_{i-1}}{p+1} - \frac{c_i}{p} + \frac{c_B}{p(p+1)}\right], \quad c_B = 0 \tag{54}$$

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Table III. Comparison between approximate analytical and numerical solutions inside cylindrical problem for small values of times.

	x				
t	0.0	0.2	0.4	0.6	0.8
	244333	160000	090000	040000	010000
0.001	247950	159799	089875	039918	009941
	241971	160000	090000	040000	010000
0.002	246041	159599	089749	039837	009883
	240152	160000	090000	040000	010000
0.003	244256	159399	089624	039755	009824
	238615	160000	090000	040000	010000
0.004	242581	159198	089498	039673	009765
	237258	160000	090000	040000	010000
0.005	241004	158998	089373	039592	009707
	231904	160000	090000	040000	010000
0.010	234238	157997	088747	039184	009413
	227766	159998	090000	040000	010000
0.015	228744	156993	088121	038776	009120
	224256	159985	090000	040000	010000
0.020	224023	155979	087496	038368	08827
	213280	159552	090000	040000	010000
0.040	208949	151697	085001	036739	007679
	210983	159318	090000	040000	010000
0.045	205717	150557	084378	036332	007400
	208803	159033	089999	040000	010000
0.050	202623	149390	083756	035925	007127

Note: The upper entry corresponds to the approximate analytical solution and the lower entry to the numerical solution. Values of $10^6 c$, $\delta t = 0.001$ and $\delta x = 0.05$.

and

$$\frac{\partial c}{\partial x} = \left(\frac{1}{\delta x}\right)\left[\frac{-pc_{i-1}}{p+1} + (p-1)\frac{c_i}{p} + \frac{c_{\mathrm{B}}}{p(p+1)}\right], \quad c_{\mathrm{B}} = 0 \tag{55}$$

where c_{B} is the concentration of the oxygen on the moving boundary.

Applying finite difference to (8), we calculate the concentration from the following equations:

$$c_{0,j+1} = 2\alpha(c_{1,j} - c_{0,j}) + c_{0,j} - \delta t, \quad \alpha = \frac{\delta t}{(\delta x)^2} \tag{56}$$

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Table IV. The values of $10^6 c$ and the position of the moving boundary for outside cylindrical problem $\delta t = 0.001$ and $\delta x = 0.1$.

	x					
t	1.1	1.3	1.5	1.7	1.9	Moving boundary
0.005	471557	282352	137285	047410	004823	1.99822
0.010	434437	275990	135720	047008	004748	1.99744
0.020	373069	255949	131267	045902	004545	1.99534
0.030	323192	231920	124215	044192	004244	1.99213
0.040	280962	207395	114948	041602	003785	1.98701
0.060	211857	161118	092749	033707	002254	1.96714
0.100	111109	084151	046006	012106	000000	1.86426
0.130	055685	038639	015154	000000	000000	1.70042
0.150	025339	013452	000000	000000	000000	1.50876
0.160	011865	002493	000000	000000	000000	1.37061
0.166	004149	000000	000000	000000	000000	1.19109
0.167	002522	000000	000000	000000	000000	1.17102

Table V. The values of $10^6 c$ and the position of the moving boundary for outside cylindrical problem $\delta t = 0.001$ and $\delta x = 0.05$.

	x					
t	1.1	1.3	1.5	1.7	1.9	Moving boundary
0.005	458655	274706	134108	046507	004982	1.99909
0.010	412439	263751	130166	045303	004797	1.99719
0.020	337164	234323	121139	042421	004232	1.99163
0.030	278037	202458	109634	038814	003471	1.98327
0.040	229662	172028	096329	034289	002545	1.97152
0.060	154565	118797	068140	022948	000480	1.93098
0.080	098936	075615	041432	010578	000000	1.85896
0.100	056408	040872	018170	000827	000000	1.74066
0.120	023287	013161	000882	000000	000000	1.54201
0.130	009524	002113	000000	000000	000000	1.36528
0.135	003293	000000	000000	000000	000000	1.21389
0.136	0.01933	000000	000000	000000	000000	1.16218

$$c_{n,j+1} = \alpha\left(1 - \frac{0.5}{1/\delta x + n}\right) c_{n-1,j} + (1-2\alpha)c_{n,j} + \alpha\left(1 + \frac{0.5}{1/\delta x + n}\right) c_{n+1,j} - \delta t, \quad n = 1, 2, 3, \ldots, (i-2) \tag{57}$$

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Table VI. Comparison between approximate analytical and numerical solutions outside cylindrical for small values of times.

	x				
t	1.0	1.2	1.4	1.6	1.8
	583507	381651	203350	086287	020721
0.001	577730	378896	202144	085879	020657
	562063	381614	203350	086287	020720
0.002	556222	376137	200931	085463	020586
	545771	381326	203350	086287	020715
0.003	538795	373373	199711	085039	020510
	532144	380586	203349	086287	020697
0.004	523928	370605	198483	084608	020425
	520219	379363	203348	086287	020660
0.005	507742	366595	197248	084170	020334
	474140	368248	203106	086277	020122
0.010	447050	343224	190888	081870	019795
	439528	353467	201822	086185	019042
0.015	399404	317353	183507	079391	019130
	410833	337887	199164	085852	017539
0.020	359475	292060	174854	076670	018353
	324780	279016	178729	079816	008489
0.040	242033	205337	134174	062196	014106
	307433	265605	172186	077011	005552
0.045	219463	187192	123788	057792	012728
	291164	252678	165330	073745	002357
0.050	198802	170266	113602	053193	011233

Note: The upper entry corresponds to the approximate analytical solution and the lower entry to the numerical solution. Values of $10^6 c$, $\delta t = 0.001$ and $\delta x = 0.05$.

and

$$c_{i-1,j+1} = \left[c_{i-1,j} + 2\alpha \left(\frac{c_{i-2,j}}{p_j+1} - \frac{c_{i-1,j}}{p_j} \right) + \frac{\delta t/\delta x \left(\frac{-p_j c_{i-2,j}}{p_j+1} + \frac{(p_j-1)c_{i-1,j}}{p_j} \right)}{(i-1)\delta x} - \delta t \right] \tag{58}$$

Finally the concentrations at the intermediate points between the two boundaries have been calculated by Equations (56) and (57), but near the moving boundary by (58). The location of the moving boundary point and conditions crossing mesh line are same as the cartesian problem

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where $p=\sqrt{2c_i}/\delta x$ and c goes on decreasing we have $c_{i,j+1\leqslant 0}$ where is physically impossible or $c_{i,j+1}>c_{i,j}$ could be caused by instability, then the formula is applied to $c_{i-1,j}$ using $p_{j-1}=p_{j-1}+1$ for more details see [10]. The results (Tables I–III for the inside problem and Tables IV–VI for the outside problem) show that an accurate solution can be obtained.

Finally for more details about Fourier–Bessel expansions and the roots of $J_0(l_s x)$, $Y_0(l_s x)$, etc. see [19–21].

7. NUMERICAL RESULTS AND DISCUSSION

All numerical calculations were performed with $\delta x=0.05$, 0.1 and $\delta t=0.001$ are given in Tables I–III for the inside cylindrical problem and Tables IV–VI for the outside cylindrical problem. We note that in this present method numerical solutions involve large errors in the beginning at the surface due to discontinuity in the gradient there at zero time, but they very soon become consistent with the approximate analytical solutions. At $t=0.020$ for the inside cylindrical problem the difference between the numerical and the approximate solutions is not more than 0.004 and at $t=0.010$ for the outside cylindrical problem is not more than 0.0044 and the movement of the boundary occurs ($\delta x=0.05$) from its original position $s=x1$, computing procedures of this present method become inoperative while the boundary is still some distance from the fixed surface $x=1$. Finally, we project the total time for complete absorption to be around 0.097 ($\delta x=0.1$), 0.085 ($\delta x=0.05$) for the inside problem and 0.167 ($\delta x=0.1$), 0.136 ($\delta x=0.05$) for the outside problem. We emphasize the importance of using the approximate analytical solution for the first few steps. We note that as the mesh size decreases, the position of the moving boundary as given by Boureghda [10] and Crank and Gupta [1] changes significantly. We therefore consider that the values as given by them are two small, and the solution has therefore a singularity at the point $x=1$, $t=0$ as well as a moving boundary.

8. CONCLUSIONS

The mathematical model of the problem has two steps. In the first step, the stable case having no oxygen transition in the isolated cylindrically shaped section is studied while in the second step the moving boundary problem of oxygen absorbed by the tissues in the cylindrically shaped section is studied, this important problem has a wide range of medical applications. In this present numerical method we describe the diffusion of oxygen in absorbing cylindrically shaped sections of tissue using the Taylor series near the moving boundary in space direction. The computation procedure showed that the present method is easy to handle with minimum error. In order to prevent the oxygen concentration from taking negative values, which causes instability, the method often resorts to small time steps. Furthermore, the numerical procedures cannot be used up to the end of absorption process due to the lack of necessary mesh points when the moving boundary is close to the sealed surface. The present paper is concerned mainly with the second stage, subsequently, the main object is to trace the moving boundary as well as the concentration at the fixed surface.

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